*Int. J. Solids Structures* Vol. 24, No. 12, pp. 1225–1234, 1988
Printed in Great Britain.
0020-7683/88 \$3.00 + .00


# NON-LINEAR FLEXURAL–FLEXURAL–TORSIONAL–EXTENSIONAL DYNAMICS OF BEAMS—I. FORMULATION

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(*Received* 13 *January* 1988; *in revised form* 20 *May* 1988)

**Abstract**—The non-linear differential equations of motion, and boundary conditions, for Euler–Bernoulli beams able to experience flexure along two principal directions (and, thus, flexure in any direction in space), torsion and extension are formulated. The beam's material is assumed to be Hookean but its properties may vary along its span. The nonlinearities present in the differential equations include contributions from the curvature expression and from inertia terms. A set of differential equations with polynomial nonlinearities to cubic order, suitable for a perturbation analysis of the motion, is also developed and the validity of the inextensional approximation is assessed. The equations developed here reduce to those for an inextensional beam. In Part II of this paper, a specific example of application is analyzed and the results obtained are compared with those available in the literature where several non-linear terms have been neglected *a priori*.

## INTRODUCTION

Many important engineering structures can be modeled as a slender member, beam-like, continuous system. Unless external constraints are imposed to restrict their motion, such structures are able to undergo flexure in any direction in space, torsion and extension. For "small" motions, it is very common to linearize the differential equations of motion of the system in order to predict its response to external excitations. In the linearization process, the bending curvatures of the element are approximated as the second spatial derivatives of its elastic bending deflections. Well-known uncoupled differential equations of motion are then obtained. For finite motions, such equations may yield a very poor approximation for the system's response. They may even yield results that are completely erroneous, for the nonlinearities may play an essential role in determining the system's response. To address such problems, it is then essential to devote special attention to the formulation of the non-linear differential equations of motion of such systems and to determine under which conditions the nonlinearities in the equations can significantly affect the motion.

It is common practice to approximate fixed–sliding or fixed–free elements as inextensional members. The non-linear differential equations describing the flexural–flexural–torsional dynamics of such elements were formulated previously (Crespo da Silva and Glynn, 1978a). Those equations are valid for arbitrary stiffness and mass variations along the beam's span. They are also valid for the general case where the bending motions are of the same order as the torsional motion. A number of cases involving the non-linear non-planar free and forced response of inextensional beams when the torsional natural frequencies are much higher than the bending natural frequencies have been analyzed (Crespo da Silva and Glynn, 1978b, 1979a, b; Crespo da Silva, 1978a, b, 1980a, b). The non-planar motions of extensional beams were considered by Ho, Scott and Eisley (1975, 1976) by making use of a set of differential equations where the bending curvature was linearized and torsion was neglected. It has been common practice in the literature to neglect a number of nonlinearities in the equations of motion of extensional beams such as non-linear contributions to the curvature and other geometric nonlinearities (Abdel-Rohman and Nayfeh, 1987; Nayfeh, 1973, 1984; Nayfeh, Mook and Lobitz, 1974a; Tezak, Mook and Nayfeh, 1978).

From a fundamentally rigorous point of view, the inextensional assumption and the differential equations of motion for fixed–sliding or for fixed–fixed boundary conditions should be a by-product of a unified approach that treats both extensional and inextensional systems. One then could assess the validity of neglecting non-linear terms, such as higher order contributions to the bending curvatures and the torsion terms, when analyzing the non-linear response of such systems.

In this paper the differential equations of motion and boundary conditions for Euler–Bernoulli extensional beams, suitable for a perturbation analysis of the non-linear flexural–flexural–torsional motions of either extensional or inextensional beams are derived in a unified and mathematically consistent manner. In the formulation developed here, it is assumed that the strains are small and that the material is Hookean. Thus, the nonlinearities in the differential equations of motion are geometric, such as those due to higher order terms in the expressions for the beam curvature vector and inertial terms. The differential equations of motion are formulated in Part I of this work. In Part II, a specific analysis of the response of a beam is analyzed and the effect of the different nonlinearities in the differential equations of motion are assessed. The equations obtained in Part I and the results obtained in Part II are compared with other work presented in the literature for more restricted situations, and the validity of several assumptions that are "usually" made for inextensional beams is discussed.

Hamiltons's extended principle is used in the formulation presented here and, thus, both the non-linear differential equations of motion and the boundary condition equations are obtained. These equations are valid for arbitrarily large motions. From these equations, a set of differential equations where all the nonlinearities are expanded to order $\varepsilon$, where $\varepsilon$ is an arbitrary parameter that is used for "bookkeeping purposes" only, are then developed. The latter equations are then suitable for a perturbation analysis of the flexural–flexural–torsional–extensional motions of the beam with arbitrary stiffness variation. This extends the work presented previously (Crespo da Silva and Glynn, 1978a) and unifies it with the work on extensional beams that has appeared in the literature to date. The Computerized Symbolic Manipulator MACSYMA (Rand, 1984) is used to perform most of the "algebraic" steps in this paper.

## KINEMATICS

Consider an initially straight undeformed thin beam of length $L$, with arbitrary boundary conditions, mass $m$ per unit length and of closed cross section. Figure 1 shows a beam segment before and after deformation. Before deformation, the length of an infinitesimal segment MN along a reference line of the beam (which defines the inertial direction $x$ shown in Fig. 1) is $dx$. After deformation, points M and N move to M* and N*, respectively, and the length of the segment M*N* is $dr$. The unit vectors $\hat{y}$ and $\hat{z}$ are inertial and normal to $\hat{x}$. Let the components, along $(\hat{x}, \hat{y}, \hat{z})$, of the elastic deformation at point M* be denoted by $u(x, t)$, $v(x, t)$ and $w(x, t)$, respectively, where $t$ is time. The orthogonal axes $(\xi, \eta, \zeta)$ centered at M* and with unit vectors $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$ are fixed to the beam's cross section normal to the reference line at M*. When the beam is undeformed the triad $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$ is aligned with $(\hat{x}, \hat{y}, \hat{z})$.

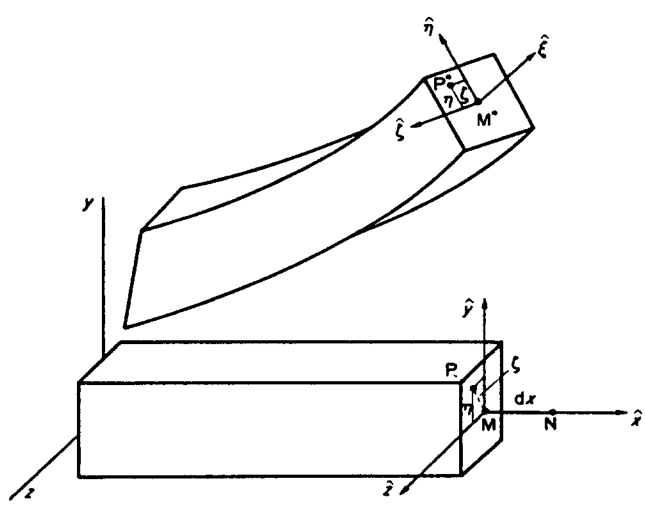


Fig. 1. Beam segment before and after deformation, and unit vector triads.

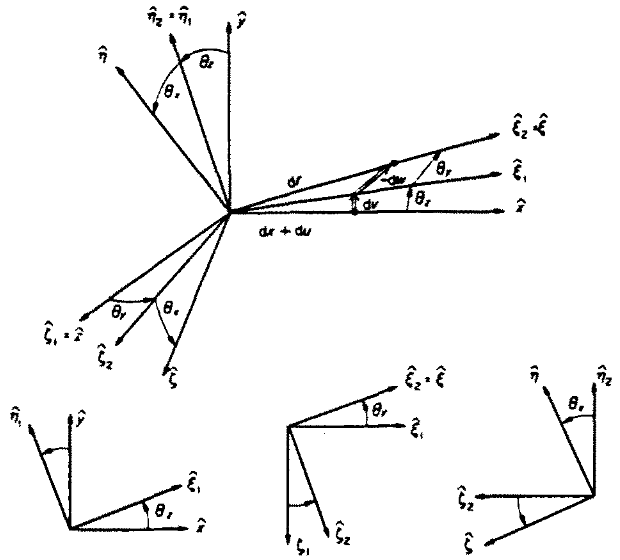

Fig. 2. Rotational sequence used to describe the orientation of the cross section axes $(\xi, \eta, \zeta)$.

The orientation of the cross section axes $(\xi, \eta, \zeta)$ relative to the reference axes $(x, y, z)$ may be described by three successive rotations. There are a total of 24 different sets of rotational sequences to describe the orientation of a body in space (Kane, Likins and Levinson, 1983). The differential equations of motion obtained for each of these 24 sequences, although "different looking", are equivalent to each other since the angles for each sequence can be related to the angles for the remaining 23 sequences by a non-linear transformation. Here (as in Crespo da Silva and Glynn (1978a)), the three-axes sequence $(\theta_z, \theta_y, \theta_x)$ shown in Fig. 2 is used to describe the orientation of the $(\xi, \eta, \zeta)$ axes in space. The relations between the elastic deflections of the reference point M* and the orientation angles $\theta_z$ and $\theta_y$ are relatively simple for that sequence. Starting by aligning the triads $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$ and $(\hat{x}, \hat{y}, \hat{z})$, the first rotation $\theta_z$ about $\hat{z}$ takes $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$ to $(\hat{\xi}_1, \hat{\eta}_1, \hat{\zeta}_1 = \hat{z})$. The second rotation $\theta_y$ about $\hat{\eta}_1$ takes $(\hat{\xi}_1, \hat{\eta}_1, \hat{\zeta}_1)$ to $(\hat{\xi}_2, \hat{\eta}_2 = \hat{\eta}_1, \hat{\zeta}_2)$ and the final rotation $\theta_x$ about $\hat{\xi}_2$ takes $(\hat{\xi}_2, \hat{\eta}_2, \hat{\zeta}_2)$ to its final orientation $(\hat{\xi} = \hat{\xi}_2, \hat{\eta}, \hat{\zeta})$. For the sake of clarity, each of the three individual rotations are shown in Fig. 1. The orientation of segment M*N* after each rotation is also shown in that figure. By letting primes denote partial differentiation with respect to $x$, it is readily seen that the no-shear assumption implies the following relations between the orientation angles $\theta_z$ and $\theta_y$ and the spatial derivatives of the displacements of point M*:

$$\tan \theta_z = v'/(1+u') \tag{1a}$$

$$\tan \theta_y = -w'/\sqrt{((1+u')^2+v'^2)}. \tag{1b}$$

Also, the elongation $e_0 \triangleq \partial r/\partial x - 1$ of the reference line at M* is obtained as

$$e_0 = \sqrt{((1+u')^2+v'^2+w'^2)} - 1. \tag{1c}$$

To obtain the differential equations of motion for the beam, the expression for the absolute angular velocity $\omega$ of the axis system $(\hat{\xi}, \hat{\eta}, \hat{\zeta})$ fixed to the cross section is needed. By letting dots denote partial differentiation with respect to time, the following expression is obtained by inspection of Fig. 2 (the symbol $\triangleq$ used here denotes "equal to by definition"):

$$\begin{aligned} \omega &= \dot{\theta}_z \hat{z} + \dot{\theta}_y \hat{\eta}_1 + \dot{\theta}_x \hat{\xi} \\ &= (\dot{\theta}_x - \dot{\theta}_z \sin \theta_y)\hat{\xi} + (\dot{\theta}_z \cos \theta_y \sin \theta_x + \dot{\theta}_y \cos \theta_x)\hat{\eta} \\ &\quad + (\dot{\theta}_z \cos \theta_y \cos \theta_x - \dot{\theta}_y \sin \theta_x)\hat{\zeta} \\ &\triangleq \omega_\xi \hat{\xi} + \omega_\eta \hat{\eta} + \omega_\zeta \hat{\zeta}. \end{aligned} \tag{2a}$$

Similarly, one can also obtain for $\mathbf{C} = \theta_z'\hat{z} + \theta_y'\hat{\eta}_1 + \theta_x'\hat{\xi}$, which is needed to generate the expression for the strain components

$$\begin{aligned}\mathbf{C} &= (\theta_x' - \theta_z' \sin\theta_y)\hat{\xi} + (\theta_z' \cos\theta_y \sin\theta_x + \theta_y' \cos\theta_x)\hat{\eta} \\ &\quad + (\theta_z' \cos\theta_y \cos\theta_x - \theta_y' \sin\theta_x)\hat{\zeta} \\ &\triangleq C_\xi\hat{\xi} + C_\eta\hat{\eta} + C_\zeta\hat{\zeta}.\end{aligned} \tag{2b}$$

The strains at any point P* of the beam's cross section are obtained from the expressions for the position vectors of P* before and after deformation. Before deformation, the position vector of P* = P is simply

$$r_P = x\hat{x} + \eta\hat{y} + \zeta\hat{z}. \tag{3a}$$

Neglecting in-plane cross section distortion and shear, point P* experiences a small axial displacement relative to M* due to warping. With this displacement given as $f(\eta, \zeta)C_\xi\hat{\xi}$, where $f(\eta, \zeta)$ is obtained by solving Laplace's equation for the cross section (Timoshenko and Goodier, 1970; Shames and Dym, 1985), the position vector of P* is then

$$r_{P^*} = (x+u)\hat{x} + v\hat{y} + w\hat{z} + \eta\hat{\eta} + \zeta\hat{\zeta} + C_\xi f(\eta, \zeta)\hat{\xi}. \tag{3b}$$

Using Green's strain measure (Shames and Dym, 1985; Annigeri, Cassenti and Dennis, 1985), one can define the strain components $\varepsilon_{ij}$ in terms of the undeformed coordinates as

$$\mathrm{d}r_{P^*} \cdot \mathrm{d}r_{P^*} - \mathrm{d}r_P \cdot \mathrm{d}r_P = 2[\mathrm{d}x, \mathrm{d}\eta, \mathrm{d}\zeta]\,[\varepsilon_{ij}] \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}\eta \\ \mathrm{d}\zeta \end{bmatrix} \quad (i = x, \eta, \zeta) \tag{4}$$

where

$$\begin{aligned}\mathrm{d}r_{P^*} &= [(1+u')\hat{x} + v'\hat{y} + w'\hat{z}]\,\mathrm{d}x + \hat{\eta}\,\mathrm{d}\eta + \hat{\zeta}\,\mathrm{d}\zeta \\ &\quad + C_\xi[(\partial f/\partial\eta)\,\mathrm{d}\eta + (\partial f/\partial\zeta)\,\mathrm{d}\xi] + \mathbf{C} \otimes [\eta\hat{\eta} + \zeta\hat{\zeta} + C_\xi f(\eta, \zeta)\hat{\xi}]\,\mathrm{d}x.\end{aligned} \tag{5}$$

In eqn (5) $\otimes$ denotes the cross product.

From eqns (3a), (4) and (5), the strain components $\varepsilon_{xx}$, $\varepsilon_{x\eta}$, $\varepsilon_{x\zeta}$ and $\varepsilon_{\eta\zeta}$ are obtained as given in eqns (6a)–(6d)

$$\varepsilon_{xx} = \{(1+e^*)^2 - 1 + [(\eta - fC_\eta)^2 + (fC_\zeta - \zeta)^2]C_\xi^2\}/2 \tag{6a}$$

$$\varepsilon_{x\eta} = [(1+e^*)\,\partial f/\partial\eta + fC_\zeta - \zeta]C_\xi/2 \tag{6b}$$

$$\varepsilon_{x\zeta} = [(1+e^*)\,\partial f/\partial\zeta + \eta - fC_\eta]C_\xi/2 \tag{6c}$$

$$\varepsilon_{\eta\zeta} = (\partial f/\partial\eta)(\partial f/\partial\zeta)C_\xi^2/2 \tag{6d}$$

where

$$e^* = \partial r/\partial x - 1 + \zeta C_\eta - \eta C_\zeta = e_0 + \zeta C_\eta - \eta C_\zeta.$$

## EQUATIONS OF MOTION

The differential equations of motion for the beam can be obtained from a vectorial approach, by using Newton's second law, or, equivalently, from a variational approach by using Hamilton's principle. The use of either approach has often been a matter of preference. Here the variational approach is used to obtain the equations of motion and the general expression for the boundary conditions associated with them. Letting $Q_\alpha$ denote the generalized forces associated with the virtual displacements $\delta\alpha$ ($\alpha = u, v, w, \theta_x$), $T$ and $U$ denote, respectively, the specific (i.e. per unit length) kinetic and strain energies associated with the motion, and $\delta W_B$ denotes the virtual work associated with forces applied at the boundaries at $x = 0$ or $L$, the extended form of Hamilton's principle (Meirovitch, 1967) then yields

$$\delta I = \int_{t=t_1}^{t_2}\int_{x=0}^{L}[\delta(T-U)+Q_u\,\delta u+Q_v\,\delta v+Q_w\,\delta w+Q_{\theta_x}\,\delta\theta_x]\,\mathrm{d}x\,\mathrm{d}t+\int_{t_1}^{t_2}\delta W_B\,\mathrm{d}t = 0. \tag{7}$$

The specific kinetic energy of motion is given as

$$\begin{aligned} T &= \frac{1}{2}\iint_A \rho(\mathrm{d}r_{P^*}/\mathrm{d}t)\cdot(\mathrm{d}r_{P^*}/\mathrm{d}t)\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}x \\ &= \frac{1}{2}\iint_A \rho\{\dot u^2+\dot v^2+\dot w^2+(\zeta\omega_\eta-\eta\omega_\zeta)^2+(\eta^2+\zeta^2)\omega_\xi^2 \\ &\quad +2(\dot u\hat x+\dot v\hat y+\dot w\hat z)\cdot[(\zeta\omega_\eta-\eta\omega_\zeta)\hat\xi-\zeta\omega_\xi\hat\eta+\eta\omega_\xi\hat\zeta]\}\,\mathrm{d}\eta\,\mathrm{d}\zeta\,\mathrm{d}x \end{aligned} \tag{8}$$

where $\rho = \rho(\eta, \zeta, x)$ is the material density at point P, and $A$ is the area of the cross section at M (Fig. 1).

If the $(\xi, \eta, \zeta)$ axes are chosen to be the principal axes of inertia of the cross section at $x = x$, and centered at the cross section's center of mass, the expression for the specific kinetic energy is reduced to the simpler form

$$T = m(\dot u^2+\dot v^2+\dot w^2)/2+(j_\xi\omega_\xi^2+j_\eta\omega_\eta^2+j_\zeta\omega_\zeta^2)/2 \tag{9}$$

where the distributed mass $m(x)$ and the distributed mass moments of inertia $j_\alpha(x)$ ($\alpha = \xi, \eta, \zeta$) are given as

$$\begin{aligned} m(x) &= \iint_A \rho(\eta,\zeta,x)\,\mathrm{d}\eta\,\mathrm{d}\zeta \\ j_\eta(x) &= \iint_A \zeta^2\rho(\eta,\zeta,x)\,\mathrm{d}\eta\,\mathrm{d}\zeta; \quad j_\zeta(x) = \iint_A \eta^2\rho(\eta,\zeta,x)\,\mathrm{d}\eta\,\mathrm{d}\zeta; \\ j_\xi(x) &= j_\eta(x)+j_\zeta(x). \end{aligned} \tag{10a-d}$$

Neglecting the normal stress components $\sigma_{\eta\eta}$ and $\sigma_{\zeta\zeta}$, and making use of engineering strains, the specific strain energy is approximated as

$$U \approx \frac{1}{2}\iint_A [\sigma_{xx}\varepsilon_{xx}+2(\sigma_{x\eta}\varepsilon_{x\eta}+\sigma_{x\zeta}\varepsilon_{x\zeta}+\sigma_{\eta\zeta}\varepsilon_{\eta\zeta})]\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{11}$$

Here it is assumed that the strains are infinitesimally small so that the linear stress–strain relationship for the material is valid. Thus, neglecting the Poisson effect, one has

$\sigma_{xx} \approx E^*\varepsilon_{xx}$, $\sigma_{xi} = G^*\varepsilon_{xi}$ ($i = \eta, \zeta$), $\sigma_{\eta\zeta} = G^*\varepsilon_{\eta\zeta}$, where $E^*(\eta, \zeta, x)$ and $G^*(\eta, \zeta, x)$ are, respectively, Young's and shear moduli for the material. Material nonlinearities are not considered here and, consistent with linear elasticity and the engineering approximations implicit in eqn (11), the engineering strains are obtained by neglecting non-linear higher order terms in eqns (6a)–(6d). These approximations have been recently addressed by Danielson and Hodges (1987) and by Bauchau and Hong (1988). Here eqns (6a)–(6d) are linearized in the elongation $e^*$. In addition, the axial displacement due to warping—and, thus, the function $f(\eta, \zeta)$ and its derivatives—is assumed to be infinitesimally small so that eqns (6a)–(6d) are also linearized in $f(\eta, \zeta)$ and its derivatives. Furthermore, the terms $fC_\eta$ and $fC_\zeta$ are also neglected in those equations. This is equivalent to accounting for the effect of warping only in the calculation of the torsional stiffness of the beam (Hodges and Dowell, 1974) and approximating $\varepsilon_{x\eta}$ and $\varepsilon_{x\zeta}$ by their dominant terms. With these simplifications, the engineering strains obtained from eqns (6) are then

$$\varepsilon_{xx} \approx e^* + (\eta^2 + \zeta^2)C_\xi^2/2 \tag{12a}$$

$$\varepsilon_{x\eta} \approx (\partial f/\partial\eta - \zeta)C_\xi/2 \tag{12b}$$

$$\varepsilon_{x\zeta} \approx (\partial f/\partial\zeta + \eta)C_\xi/2 \tag{12c}$$

$$\varepsilon_{\eta\zeta} \approx 0. \tag{12d}$$

Making use of eqns (12a)–(12d), the expression for the specific strain energy then becomes

$$U = \{EAe_0^2 + D_\eta C_\eta^2 + D_\zeta C_\zeta^2 + [D_\xi + e_0(D_\eta + D_\zeta)]C_\xi^2\}/2 + \{e_0(e_\zeta C_\eta - e_\eta C_\zeta) - e_{\eta\zeta}C_\eta C_\zeta\}$$
$$+ \left\{\tfrac{1}{2}C_\xi^2 \iint_A E^*(\zeta C_\eta - \eta C_\zeta)(\eta^2 + \zeta^2)\, d\eta\, d\zeta + \tfrac{1}{8}C_\xi^4 \iint_A E^*(\eta^2 + \zeta^2)^2\, d\eta\, d\zeta\right\} \tag{13}$$

where

$$E(x) = \iint_A E^*(\eta, \zeta, x)\, d\eta\, d\zeta/A(x), \quad A(x) = \iint_A d\eta\, d\zeta$$

$$D_\eta(x) = \iint_A E^*\zeta^2\, d\eta\, d\zeta, \quad D_\zeta(x) = \iint_A E^*\eta^2\, d\eta\, d\zeta$$

$$D_\xi(x) = \iint_A G^*[(\eta + \partial f/\partial\zeta)^2 + (\zeta - \partial f/\partial\eta)^2]\, d\eta\, d\zeta$$

$$e_\zeta(x) = \iint_A E^*\zeta\, d\eta\, d\zeta, \quad e_\eta(x) = \iint_A E^*\eta\, d\eta\, d\zeta$$

$$e_{\eta\zeta}(x) = \iint_A E^*\eta\zeta\, d\eta\, d\zeta. \tag{14a–h}$$

$D_\eta$, $D_\zeta$ and $D_\xi$ are, respectively, the bending and torsional stiffnesses of the beam. For beams where Young's modulus is only a function of $x$, $E^* = E(x)$ in eqn (13). Also, if the material density along the beam is only a function of $x$, the mass center and the area centroid of the cross section at $x = x$ coincide with each other. For this case, all the terms in the second bracketed group in eqn (13) vanish since $e_\zeta = e_\eta = e_{\eta\zeta} = 0$. The terms in the third bracketed group in eqn (13) involve higher order area cross section integrals and their contribution to the differential equations of motion are neglected. For uniform and inextensional beams, where $E^* = E(x)$, $\rho = \rho(x)$ and $e_0 = 0$, the expression for the strain energy given by eqn (13) reduces to the simple form given previously (Crespo da Silva and Glynn, 1978a).

By taking the variations of the kinetic and strain energies given by eqns (9) and (13), and by integrating by parts some of the terms resulting from eqn (7), the differential equations of motion for the beam can be put in the same form as those given previously (Crespo da Silva and Glynn, 1978a) as

$$G_u' = [A_{\theta_z}\,\partial\theta_z/\partial u' + A_{\theta_y}\,\partial\theta_y/\partial u' + \lambda(1+u')]' = m\ddot{u} - Q_u \tag{15a}$$

$$G_\alpha' = [A_{\theta_z}\,\partial\theta_z/\partial\alpha' + A_{\theta_y}\,\partial\theta_y/\partial\alpha' + \lambda\alpha']' = m\ddot{\alpha} - Q_\alpha \quad (\alpha = v, w) \tag{15b, c}$$

$$A_{\theta_x} = Q_{\theta_x} \tag{15d}$$

where

$$\lambda = EAe_0/(1+e_0) \tag{16a}$$

and, with $l = T - U + EAe_0^2/2$

$$A_\alpha = \frac{\partial^2 l}{\partial\dot{\alpha}\,\partial t} + \frac{\partial^2 l}{\partial\alpha'\,\partial x} - \frac{\partial l}{\partial\alpha} \quad (\alpha = \theta_z, \theta_y, \theta_x). \tag{16b–d}$$

For inextensional beams, $\lambda$ stands for a Lagrange multiplier which can be determined as illustrated previously (Crespo da Silva and Glynn, 1978b).

The terms that are integrated by parts in eqn (7) yield the following boundary condition equation:

$$\left[\frac{\partial l}{\partial\theta_x'}\delta\theta_x - G_u\,\delta u - G_v\,\delta v - G_w\,\delta w + H_u\,\delta u' + H_v\,\delta v' + H_w\,\delta w' + \delta W_B\right]_{x=i} = 0 \quad (i = 0, L) \tag{17a, b}$$

where

$$H_\alpha = \frac{\partial l}{\partial\theta_y'}\frac{\partial\theta_y}{\partial\alpha'} + \frac{\partial l}{\partial\theta_z'}\frac{\partial\theta_z}{\partial\alpha'} \quad (\alpha = u, v, w). \tag{17c–e}$$

## EQUATIONS OF MOTION EXPANDED TO $O(\varepsilon^3)$ NONLINEARITIES

The partial differential equations, eqns (15a)–(15d) are nonlinear and coupled. They are valid for arbitrarily large deformations as long as the stresses in the material are linearly related to the engineering strains. To be able to analyze the motion by perturbation techniques, the beam deformations are now restricted in magnitude so that all nonlinearities in those equations are expanded in Taylor series about an equilibrium solution which is here taken to be $u = v = w = \theta_x = 0$. This can be accomplished by first expressing the elastic deformations $u$, $v$, $w$ and $\theta_x$ in terms of a small arbitrary ordering parameter $\varepsilon$ as $\alpha(x,t) = \varepsilon\alpha_1(x,t)$ $(\alpha = v, w, \theta_x)$, $u(x,t) = \varepsilon^2 u_2(x,t)$, and then expanding eqns (15a)–(15d) in a Taylor series in $\varepsilon$. We also let $EA = (EA)_2/\varepsilon$ so that the terms $\lambda v'$ and $\lambda w'$ in eqns (15b) and (15c) are transferred from the linearized $O(\varepsilon)$ equations to the next higher order approximation. The expansion of equations was done by computer with MACSYMA (Rand, 1984). Letting $Q_\alpha^*(x,t)$ denote the expanded form of $Q_\alpha(\alpha = u, v, w, \theta_x)$, and dropping the subscripted and starred notations for convenience, the $O(\varepsilon^3)$ differential equations of motion are obtained as given in eqns (18a)–(18d)

$$G_u' = \{w'(D_\eta w'' + e_{\eta\zeta} v'')' + v'(D_\zeta v'' + e_{\eta\zeta} w'')' - j_\eta w' \ddot{w}' - j_\zeta v' \ddot{v}' + \lambda\}' + O(\varepsilon^3) = m\ddot{u} - Q_u \quad (18a)$$

$$\begin{aligned} G_v' = \{ & -(D_\zeta v'' + e_{\eta\zeta} w'')' + j_\zeta \ddot{v}' + [(D_\eta - D_\zeta)(\theta_x w'' - \theta_x^2 v'') \\ & - D_\zeta w'(\theta_x' + v'' w')]' + v'(D_\zeta e_0')' + 2(D_\zeta v'' e_0)' - D_\zeta v'(v''^2 + w''^2) \\ & + v' w'[(D_\eta - D_\zeta) w'']' + [e_0(e_\eta - e_\zeta \theta_x) + e_{\eta\zeta}(2\theta_x v'' + w'(u' + v'^2)' \\ & + (2u' + 3v'^2/2 + 3w'^2/2 + 2\theta_x^2) w'')]' - e_{\eta\zeta} w''(u' + v'^2)' - e_{\eta\zeta} v''(v' w')' \\ & + j_\zeta [w'(\dot{\theta}_x + \dot{v}' w')]^{\cdot} - (j_\eta - j_\zeta)(\theta_x \dot{w}' - \theta_x^2 \dot{v}')^{\cdot} - j_\zeta v' \ddot{e}_0 - 2 j_\zeta (\dot{v}' e_0)^{\cdot} \\ & + j_\zeta v'(\dot{v}'^2 + \dot{w}'^2) + (j_\zeta - j_\eta) v' w' \ddot{w}' + \lambda v'\}' + O(\varepsilon^4) = m\ddot{v} - Q_v \end{aligned} \quad (18b)$$

$$\begin{aligned} G_w' = \{ & -(D_\eta w'' + e_{\eta\zeta} v'')' + j_\eta \ddot{w}' + [(D_\eta - D_\zeta)(\theta_x v'' + \theta_x^2 w'')]' \\ & - D_\eta w' w''^2 + w'(D_\eta e_0')' + 2e_0(D_\eta w'')' + D_\zeta v''(\theta_x' + v'' w') - D_\zeta w' v''^2 \\ & + [e_0(e_\zeta + e_\eta \theta_x)]' + (e_{\eta\zeta} v'')' e_0 - [e_{\eta\zeta}(2\theta_x w'' - (u' v')' - (v'^2 + w'^2 + 2\theta_x^2) v'')]' \\ & - 2 e_{\eta\zeta} w' v'' w'' - j_\zeta \dot{v}'(\dot{\theta}_x + \dot{v}' w') + (j_\zeta - j_\eta)[(\theta_x \dot{v}' + \theta_x^2 \dot{w}')^{\cdot} + \dot{v}'^2 w'] \\ & - j_\eta [(2u' + v'^2 + 2w'^2) \ddot{w}' + 2\dot{w}' \dot{e}_0 + w'(\ddot{u}' + v' \ddot{v}')] + \lambda w'\}' + O(\varepsilon^4) = m\ddot{w} - Q_w \end{aligned} \quad (18c)$$

$$\begin{aligned} A_{\theta_x} = & -[D_x(\theta_x' + v'' w') + D_{\eta\zeta} \theta_x' e_0]' + (D_\eta - D_\zeta)[(v''^2 - w''^2)\theta_x - v'' w''] \\ & + (e_\zeta v'' - e_\eta w'')(v'^2 + w'^2)/2 + e_{\eta\zeta}(w''^2 - v''^2 - 4\theta_x v'' w'') \\ & + j_\zeta(\dot{\theta}_x + w' \dot{v}')^{\cdot} + (j_\zeta - j_\eta)[(\dot{v}'^2 - \dot{w}'^2)\theta_x - \dot{v}' \dot{w}'] + O(\varepsilon^4) = Q_{\theta_x} \end{aligned} \quad (18d)$$

where $e_0 = u' + (v'^2 + w'^2)/2$ and $\lambda = EAe_0$.

Equations (18a)–(18d) are valid for arbitrary property variation along the beam's span and for arbitrary boundary conditions. In particular, if the material density along the beam is only a function of $x$, then $e_\eta = e_\zeta = e_{\eta\zeta} = 0$. It can be readily verified that for inextensional beams, where $u' = -(v'^2 + w'^2)/2 + O(\varepsilon^4)$, those equations reduce to eqns (11a)–(11d) in Crespo da Silva and Glynn (1978a). As indicated by eqns (18b) and (18c), the expression for $u'(x, t)$, which can be obtained from eqn (18a), is only needed to $O(\varepsilon^2)$. The expanded form of the functions $H_u$, $H_v$ and $H_w$ that appear in the boundary conditions, eqns (17a) and (17b), are also given below. To $O(e^3)$, $\partial l/\partial\theta_x' = -D_\xi(\theta_x' + v'' w')$

$$H_u = D_\eta w' w'' + D_\zeta v' v'' + e_{\eta\zeta}(v' w')' + O(\varepsilon^3) \quad (19a)$$

$$\begin{aligned} H_v = & -D_\xi(\theta_x' + v'' w') w' - (D_\eta - D_\zeta)[(v'' \theta_x - w'')\theta_x - v' w' w''] \\ & - D_\zeta[v'' - v' e_0' - 2v'' e_0] + (e_\eta - e_\zeta \theta_x) e_0 \\ & + e_{\eta\zeta}\{2\theta_x(v'' + \theta_x w'') + [w'(u' + v'^2)]' + w''(u'' - v'^2/2 + 3w'^2/2)\} + O(\varepsilon^4) \end{aligned} \quad (19b)$$

$$\begin{aligned} H_w = & (D_\eta - D_\zeta)(v'' + \theta_x w'')\theta_x - D_\eta[w'' - w' e_0' - 2w'' e_0] + (e_\zeta + e_\eta \theta_x) e_0 \\ & + e_{\eta\zeta}[2\theta_x(v'' \theta_x - w'') - v'' + (u' v')' + v''(u'' + 3v'^2/2 + 3w'^2/2)] + O(\varepsilon^4). \end{aligned} \quad (19c)$$

## VALIDITY OF THE INEXTENSIONAL APPROXIMATION

The validity of the inextensional approximation can be assessed by first integrating eqn (18a) from $x = L$ to $x$ to obtain

$$G_u(x,t) = G_u(L,t)+\int_L^x [m\ddot{u}(y,t)-Q_u(y,t)]\,\mathrm{d}y \equiv EA\left(u'+\frac{v'^2+w'^2}{2}\right)+O(\varepsilon^2). \quad (20)$$

To $O(\varepsilon^2)$, and with $Q_u(x,t) = O(\varepsilon)$, eqn (20) can be solved for $u'(x,t)$ as

$$u'(x,t) = \left[G_u(L,t)-\int_L^x Q_u(y,t)\,\mathrm{d}y\right]\Big/EA-(v'^2+w'^2)/2+O(\varepsilon^3). \quad (21)$$

To $O(\varepsilon^3)$, the inextensionality condition $e_0 = 0$ becomes $u' = -(v'^2+w'^2)/2$. As indicated by eqn (21), the inextensionality condition is satisfied when $G_u(L,t) = 0$ and $Q_u(x,t) = 0$. When $G_u(L,t) = 0$ and the beam is subjected to a force with an $x$ component (Crespo da Silva, 1978a, b), the inextensionality condition is approached as $EA \to \infty$. The limiting approach $EA \to \infty$ was used by Hodges, Ormiston and Peters (1980) when considering the kinematics of a rotating beam as an example in their work on the non-linear deformation geometry of Euler–Bernoulli beams. The condition $G_u(L,t) = 0$ is satisfied for free–free, fixed–free of fixed–sliding boundaries.

## CONCLUDING REMARKS

In this paper, the non-linear differential equations of motion for Euler–Bernoulli beams undergoing extension, flexure along two principal directions, and torsion, have been formulated. Unlike other formulations presented in the literature, the effects of all geometric nonlinearities, which arise from midplane stretching, curvature and inertia terms, have been considered. The equations are valid for arbitrary stiffness and mass variation along the beam's span. A set of $O(\varepsilon^3)$ differential equations, suitable for a perturbation analysis of the motion, has also been developed. Here $\varepsilon$ is an arbitrary perturbation parameter that is used for "bookkeeping purposes" only, and the flexural and torsional elastic deformations are taken to be of $O(\varepsilon)$. The equations developed here reduce to those for an inextensional beam (Crespo da Silva and Glynn, 1978a) by taking into account all the geometric nonlinearities.

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